

Practical tools for third order cosmological perturbations

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(Dated: November 17, 2009)

We discuss cosmological perturbation theory at third order, deriving the gauge transformation rules for metric and matter perturbations, and constructing third order gauge invariant quantities. We present the Einstein tensor components, the evolution equations for a perfect fluid, and the Klein-Gordon equation at third order, including scalar, vector and tensor perturbations. In doing so, we also give all second order tensor components and evolution equations in full generality.

PACS numbers: 98.80.Jk, 98.80.Cq

arXiv:0909.0942

I. INTRODUCTION

Anisotropies in the cosmic microwave background and large scale structure provide compelling evidence that the universe is not truly homogeneous and isotropic and therefore cannot be described completely by the Friedmann-Robertson-Walker (FRW) spacetime. Since general relativity is highly nonlinear, it is extremely difficult to find exact solutions which would allow for all the inhomogeneities and anisotropies present in the real universe. Thus one usually resorts to perturbative techniques, considering a homogeneous FRW background and adding inhomogeneous perturbations. This is the basis of cosmological perturbation theory, which has become a cornerstone of modern cosmology in the last half century (see for example Refs. [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13]).

Linear perturbation theory is now an essential tool in the cosmologist's toolbox, and extending the theory beyond linear order is at present a rapidly expanding area of research: recent years have seen second order theory gaining the attention it duly deserves and becoming more mature. However, one does not have to stop there: it is worthwhile and feasible to consider perturbation theory even beyond second order, and in this paper we study aspects of cosmological perturbation theory at third order.

Each order in perturbation theory reveals different though complementary aspects of the underlying fully non-linear theory, and also gives us access to information of different data sets or to extract more information from the same data set. First order theory allows us to model the large scale structure of the universe and to calculate the distribution of anisotropies in the CMB. Second order theory then enables us to access even more information from these data sets, by calculating higher order observables such as the bispectrum. Third order reveals yet even more information, by calculating, for example, the trispectrum.

Whereas at linear order the different types of perturbations, denoted by their transformation behaviour on three-hypersurfaces as scalar, vector and tensor, decouple, this is no longer the case beyond first order. At second order we find, for example in the energy conservation equation, couplings between first order scalar and vector perturbations through gradients, and also the coupling of first order tensor perturbations to each other. At third order a new coupling occurs in the energy conservation equation, namely the coupling of scalar perturbations to tensor perturbations. This will allow for the calculation of yet another different observational signature, highlighting another aspect of the underlying full theory.

There has already been some work on third order theory. For example, Ref. [14, 15] considered third order perturbations of pressureless irrotational fluids as “pure” general relativistic correction terms to second order quantities. The calculations focused on the temporal comoving gauge, allowing the authors to consider only second order geometric and energy-momentum components, and neglected vector perturbations. Ref. [16] includes a study of third order perturbations with application to the trispectrum in the two-field ekpyrotic scenario in the large scale limit. There has also been reference in the literature of the need to extend perturbation theory beyond second order. For example, in Ref. [17] UV divergences in the Raychaudhuri equation are found when considering backreaction from averaging perturbations to second order. The authors state that these divergences may be removed by extending

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perturbation theory to third, or higher, orders.

In this paper, we develop the essential tools for third order perturbation theory, such as the gauge transformation rules for different types of perturbation, and construct gauge invariant quantities at third order. We consider perfect fluids with non-zero pressure, including all types of perturbation, namely, scalar, vector and tensor perturbations. In particular allowing for vector perturbations is crucial for realistic higher order studies, as we have recently shown that vorticity is generated at second order in the perturbations in all models employing non-barotropic fluids [18]. Hence studying irrotational fluids at higher order will only give partial insight into the underlying physics. We present the energy and momentum conservation equations for such a fluid, and also give the components of the perturbed Einstein tensor, up to third order. All equations are given without fixing a gauge. We also give the Klein Gordon equation for the scalar field minimally coupled to gravity at third order in cosmological perturbation theory.

The paper is organised as follows: in the next section we give our definitions, followed by the gauge transformation rules in section III. We construct gauge invariant variables at third order in section IV, present the governing evolution and constraint equations in section V and conclude with a discussion in section VI

In this paper, we use conformal time, η , throughout, denoting derivatives with respect to conformal time with a prime. The scale factor is a , the comoving Hubble parameter is $\mathcal{H} = a'/a$ and the physical Hubble rate is $H = a'/a^2$. Greek indices, μ, ν, λ run from $0, \dots, 3$, while lower case Latin indices, i, j, k , take the value 1, 2, or 3. Covariant derivatives are denoted by a semi-colon, partial derivatives by a comma. The order of the perturbations is usually denoted with a subscript immediately after a perturbed quantity, when it is split up order by order. We assume a spatially flat FRW background throughout.

II. DEFINITIONS

The components of the covariant metric tensor are

$$g_{00} = -a^2(1 + 2\phi), \quad g_{0i} = a^2 B_i, \quad g_{ij} = a^2(\delta_{ij} + 2C_{ij}), \quad (2.1)$$

where we take the three dimensional background space to be flat with metric δ_{ij} , and include scalar, vector and tensor perturbations. Here, ϕ is the lapse function and B_i is the vector perturbation which can be split into the gradient of a scalar, $B_{,i}$, and a divergence-free vector, S_i .¹ C_{ij} is the spatial part of the metric perturbation, which can be decomposed as $C_{ij} = -\psi\delta_{ij} + E_{,ij} + F_{(i,j)} + \frac{1}{2}h_{ij}$. Note that in the uniform curvature gauge, this reduces to $C_{ij} = \frac{1}{2}h_{ij}$, where h_{ij} is the tensor, or gravitational wave, perturbation.

The contravariant metric components are obtained by imposing the constraint $g_{\mu\nu}g^{\nu\lambda} = \delta_\mu^\lambda$, to the appropriate order. To third order this gives

$$g^{00} = -\frac{1}{a^2} (1 - 2\phi + 4\phi^2 - 8\phi^3 - B_k B^k + 4\phi B_k B^k + 2B^i B^j C_{ij}), \quad (2.2)$$

$$g^{0i} = \frac{1}{a^2} (B^i - 2\phi B^i - 2B_k C^{ki} + 4\phi^2 B^i + 4B_k C^{ki}\phi + 4C^{kj} C_j^i B_k - B^k B_k B^i), \quad (2.3)$$

$$g^{ij} = \frac{1}{a^2} (\delta^{ij} - 2C^{ij} + 4C^{ik} C_k^j - B^i B^j + 2\phi B^i B^j - 8C^{ik} C^{jl} C_{kl} + 2B^i C^{kj} B_k + 2B_k B^j C^{ik}). \quad (2.4)$$

The perturbations can be split order by order as, e.g.,

$$\phi = \phi_1 + \frac{1}{2}\phi_2 + \frac{1}{3!}\phi_3, \quad (2.5)$$

but we do not explicitly perform such a split here in order to keep expressions compact. For example, expanding the $0-0$ component fully order by order gives

$$g^{00} = -\frac{1}{a^2} \left(1 - 2\phi_1 - \phi_2 - \frac{1}{3}\phi_3 + \phi_1^2 + 8\phi_1\phi_2 - 8\phi_1^3 - B_{1k} B_1^k - B_{2k} B_1^k + 4\phi_1 B_1^k B_{1k} + 2B_1^i B_1^j C_{1ij} \right), \quad (2.6)$$

¹ Note that our sign convention differs to that in Ref. [15], by a minus sign in front of B_i .

which when compared to Eq. (2.2) illustrates why we do not split perturbations where possible in this paper.

The energy momentum tensor for a perfect fluid is

$$T^\mu{}_\nu = (\rho + P)u^\mu u_\nu + P\delta^\mu{}_\nu, \quad (2.7)$$

where P is the pressure, ρ is the energy density and u^μ is the fluid four velocity, defined as

$$u^\mu = \frac{dx^\mu}{d\tau}. \quad (2.8)$$

The fluid velocity is subject to the constraint

$$u^\mu u_\mu = -1, \quad (2.9)$$

and, to third order in the perturbations, has components

$$u^i = \frac{1}{a}v^i, \quad (2.10)$$

$$u^0 = \frac{1}{a} \left(1 - \phi + \frac{3}{2}\phi^2 - \frac{5}{2}\phi^3 + \frac{1}{2}v_k v^k + v_k B^k + C_{kj}v^k v^j - 2\phi v^k B_k - \phi v^k v_k \right), \quad (2.11)$$

$$u_i = a \left(v_i + B_i - \phi B_i + 2C_{ik}v^k + \frac{3}{2}B_i\phi^2 + \frac{1}{2}B_i v^k v_k + B_i v^k B_k \right), \quad (2.12)$$

$$u_0 = -a \left(1 + \phi - \frac{1}{2}\phi^2 + \frac{1}{2}\phi^3 + \frac{1}{2}v^k v_k + \phi v_k v^k + C_{kj}v^k v^j \right). \quad (2.13)$$

III. GAUGE TRANSFORMATIONS

We use here the active approach to deal with gauge transformations, where the exponential map is the starting point (we refer the reader to the recent reviews on perturbation theory [10, 11] for detailed discussions of gauge issues and related matters in this section). This allows us to immediately write down how a tensor \mathbf{T} transforms once the generator of the gauge transformation, ξ^μ , has been specified. The exponential map is then

$$\widetilde{\mathbf{T}} = e^{\mathcal{L}_\xi} \mathbf{T}, \quad (3.1)$$

where \mathcal{L}_ξ denotes the Lie derivative with respect to ξ^λ . The vector field generating the transformation, ξ^λ , is up to third order

$$\xi^\mu \equiv \epsilon \xi_1^\mu + \frac{1}{2}\epsilon^2 \xi_2^\mu + \frac{1}{6}\epsilon^3 \xi_3^\mu + O(\epsilon^4). \quad (3.2)$$

The exponential map can be readily expanded and, taking into account Eq. (3.2), gives

$$\exp(\mathcal{L}_\xi) = 1 + \epsilon \mathcal{L}_{\xi_1} + \frac{1}{2}\epsilon^2 \mathcal{L}_{\xi_1}^2 + \frac{1}{2}\epsilon^2 \mathcal{L}_{\xi_2} + \frac{1}{6}\epsilon^3 \mathcal{L}_{\xi_3} + \frac{1}{6}\epsilon^3 \mathcal{L}_{\xi_1}^3 + \frac{1}{4}\epsilon^3 \mathcal{L}_{\xi_1} \mathcal{L}_{\xi_2} + \frac{1}{4}\epsilon^3 \mathcal{L}_{\xi_2} \mathcal{L}_{\xi_1} + \dots \quad (3.3)$$

where we have kept terms up to $O(\epsilon^3)$. Splitting the tensor \mathbf{T} up to third order, as given in Eq. (2.5), and collecting terms of like order in ϵ we find that tensorial quantities transform at zeroth, first, second, and third order, respectively, as [12, 19]

$$\widetilde{\mathbf{T}}_0 = \mathbf{T}_0, \quad (3.4)$$

$$\epsilon \delta \widetilde{\mathbf{T}}_1 = \epsilon \delta \mathbf{T}_1 + \epsilon \mathcal{L}_{\xi_1} \mathbf{T}_0, \quad (3.5)$$

$$\epsilon^2 \delta \widetilde{\mathbf{T}}_2 = \epsilon^2 (\delta \mathbf{T}_2 + \mathcal{L}_{\xi_2} \mathbf{T}_0 + \mathcal{L}_{\xi_1}^2 \mathbf{T}_0 + 2\mathcal{L}_{\xi_1} \delta \mathbf{T}_1), \quad (3.6)$$

$$\epsilon^3 \delta \widetilde{\mathbf{T}}_3 = \epsilon^3 \left\{ \delta \mathbf{T}_3 + \left(\mathcal{L}_{\xi_3} + \mathcal{L}_{\xi_1}^3 + \frac{3}{2}\mathcal{L}_{\xi_1} \mathcal{L}_{\xi_2} + \frac{3}{2}\mathcal{L}_{\xi_2} \mathcal{L}_{\xi_1} \right) \mathbf{T}_0 + 3(\mathcal{L}_{\xi_1}^2 + \mathcal{L}_{\xi_2}) \delta \mathbf{T}_1 + 3\mathcal{L}_{\xi_1} \delta \mathbf{T}_2 \right\}. \quad (3.7)$$

Applying the map (3.1) to the coordinate functions x^μ , we get the following relationship between the coordinates of a point p and a point q

$$x^\mu(q) = e^{\xi^\lambda \frac{\partial}{\partial x^\lambda}} \Big|_p x^\mu(p), \quad (3.8)$$

where we have used the fact that when acting on scalars $\mathcal{L}_\xi = \xi^\mu(\partial/\partial x^\mu)$ and the partial derivatives are evaluated at p . The left-hand-side and the right-hand-side of Eq. (3.8) are evaluated at different points. Equation (3.8) can then be expanded up to third order as

$$\begin{aligned} x^\mu(q) = & x^\mu(p) + \epsilon \xi_1^\mu(p) + \frac{1}{2} \epsilon^2 (\xi_{1,\nu}^\mu(p) \xi_1^\nu(p) + \xi_2^\mu(p)) \\ & + \frac{1}{6} \epsilon^3 \left[\xi_3^\mu(p) + \left(\xi_{1,\lambda\beta}^\mu + \xi_{1,\beta}^\mu \xi_{1,\lambda}^\beta \right) \xi_1^\lambda(p) \right] + \frac{1}{4} \epsilon^3 \left(\xi_{2,\lambda}^\mu(p) \xi_1^\lambda(p) + \xi_{1,\lambda}^\mu(p) \xi_2^\lambda(p) \right), \end{aligned} \quad (3.9)$$

relating the coordinates of the points p and q .

A. Four-scalars

We now turn to four-scalars, using here the energy density ρ as an example, and study its behaviour under a gauge transformation up to third order. It can be readily expanded as

$$\rho = \rho_0 + \delta\rho_1 + \frac{1}{2}\delta\rho_2 + \frac{1}{3!}\delta\rho_3, \quad (3.10)$$

and we shall now detail the transformation order by order.

1. First order

The Lie derivative of a scalar is simply given by

$$\mathcal{L}_\xi \rho = \xi^\lambda \rho_{,\lambda}. \quad (3.11)$$

Before we can study the transformation behaviour of the perturbations at first order, we split the generating vector ξ_1^μ into a scalar temporal part α_1 and a spatial scalar and vector part, β_1 and γ_1^i , according to

$$\xi_1^\mu = (\alpha_1, \beta_1,^i + \gamma_1^i), \quad (3.12)$$

where the vector part is divergence-free ($\partial_k \gamma_1^k = 0$).

Under a first-order transformation a four scalar, here the energy density, ρ , then transforms, from Eqs. (3.5) and (3.11) as,

$$\widetilde{\delta\rho_1} = \delta\rho_1 + \rho'_0 \alpha_1. \quad (3.13)$$

The first-order density perturbation can be made gauge-invariant by prescribing the first order temporal gauge or time slicing, α_1 .

2. Second order

At second order, as at first order, we split the generating vector ξ_2^μ into a scalar temporal and scalar and vector spatial part, as

$$\xi_2^\mu = (\alpha_2, \beta_2,^i + \gamma_2^i), \quad (3.14)$$

where the vector part is divergence-free ($\partial_k \gamma_2^k = 0$). We then find from Eqs. (3.6) and (3.11) that a four scalar transforms as

$$\widetilde{\delta\rho_2} = \delta\rho_2 + \rho'_0 \alpha_2 + \alpha_1 (\rho''_0 \alpha_1 + \rho'_0 \alpha_1' + 2\delta\rho_1') + (2\delta\rho_1 + \rho'_0 \alpha_1)_{,k} (\beta_1,^k + \gamma_1^k). \quad (3.15)$$

We now need to specify the time slicing at first and second order, and also the spatial gauge or threading at first order, in order to render the second order density perturbation gauge-invariant.

3. Third order

At third order we also split the generating vector ξ_3^μ into a scalar temporal and scalar and vector spatial part, as

$$\xi_3^\mu = (\alpha_2, \beta_2,^i + \gamma_3^i), \quad (3.16)$$

where the vector part is again divergence-free ($\partial_k \gamma_3^k = 0$). We then find from Eqs. (3.7) and (3.11) that the energy density transforms as

$$\widetilde{\delta\rho_3} = \delta\rho_3 + \left(\mathcal{L}_{\xi_3} + \mathcal{L}_{\xi_1}^3 + \frac{3}{2} \mathcal{L}_{\xi_1} \mathcal{L}_{\xi_2} + \frac{3}{2} \mathcal{L}_{\xi_2} \mathcal{L}_{\xi_1} \right) \rho_0 + 3 (\mathcal{L}_{\xi_1}^2 + \mathcal{L}_{\xi_2}) \delta\rho_1 + 3 \mathcal{L}_{\xi_1} \delta\rho_2, \quad (3.17)$$

which gives

$$\begin{aligned} \widetilde{\delta\rho_3} = & \delta\rho_3 + \rho'_0 \alpha_3 + \rho''_0 \alpha_1^3 + 3\rho''_0 \alpha_1 \alpha_{1,\lambda} \xi_1^\lambda + \rho'_0 (\alpha_{1,\lambda\beta} \xi_1^\lambda + \alpha_{1,\lambda} \xi_{1,\beta}^\lambda) \xi_1^\beta \\ & + 3\rho''_0 \alpha_1 \alpha_2 + \rho'_0 \frac{3}{2} (\alpha_{2,\lambda} \xi_1^\lambda + \alpha_{1,\lambda} \xi_2^\lambda) + 3 (\delta\rho_{1,\lambda\beta} \xi_1^\lambda + \delta\rho_{1,\lambda} \xi_{1,\beta}^\lambda) \xi_1^\beta + 3\delta\rho_{1,\lambda} \xi_2^\lambda + 3\delta\rho_{2,\lambda} \xi_1^\lambda. \end{aligned} \quad (3.18)$$

Similar to the second order case, we need to specify the time slicings (at all orders), and also the spatial gauge or threading at first and second order, in order to render the third order density perturbation gauge-invariant.

B. The metric tensor

We now give the transformation behaviour of the metric tensor. As above, we refer to Refs. [10] and [11] for details on the first and second order calculations, and focus the detailed calculation on the third order derivation. The starting point is again the Lie derivative, which for a covariant tensor is given by

$$\mathcal{L}_\xi g_{\mu\nu} = g_{\mu\nu,\lambda} \xi^\lambda + g_{\mu\lambda} \xi_{,\nu}^\lambda + g_{\lambda\nu} \xi_{,\mu}^\lambda. \quad (3.19)$$

1. First and second order

The metric tensor transforms at first order, from Eqs. (3.6) and (3.19), as

$$\widetilde{\delta g_{\mu\nu}^{(1)}} = \delta g_{\mu\nu}^{(1)} + g_{\mu\nu,\lambda}^{(0)} \xi_1^\lambda + g_{\mu\lambda}^{(0)} \xi_{1,\nu}^\lambda + g_{\lambda\nu}^{(0)} \xi_{1,\mu}^\lambda, \quad (3.20)$$

and at second order as

$$\begin{aligned} \widetilde{\delta g_{\mu\nu}^{(2)}} = & \delta g_{\mu\nu}^{(2)} + g_{\mu\nu,\lambda}^{(0)} \xi_2^\lambda + g_{\mu\lambda}^{(0)} \xi_{2,\nu}^\lambda + g_{\lambda\nu}^{(0)} \xi_{2,\mu}^\lambda + 2 \left[\delta g_{\mu\nu,\lambda}^{(1)} \xi_1^\lambda + \delta g_{\mu\lambda}^{(1)} \xi_{1,\nu}^\lambda + \delta g_{\lambda\nu}^{(1)} \xi_{1,\mu}^\lambda \right] \\ & + g_{\mu\nu,\lambda\alpha}^{(0)} \xi_1^\lambda \xi_1^\alpha + g_{\mu\nu,\lambda}^{(0)} \xi_{1,\alpha}^\alpha \xi_1^\lambda + 2 \left[g_{\mu\lambda,\alpha}^{(0)} \xi_1^\alpha \xi_{1,\nu}^\lambda + g_{\lambda\nu,\alpha}^{(0)} \xi_1^\alpha \xi_{1,\mu}^\lambda + g_{\lambda\alpha}^{(0)} \xi_{1,\mu}^\lambda \xi_{1,\nu}^\alpha \right] \\ & + g_{\mu\lambda}^{(0)} (\xi_{1,\nu\alpha}^\lambda \xi_1^\alpha + \xi_{1,\alpha}^\lambda \xi_{1,\nu}^\alpha) + g_{\lambda\nu}^{(0)} (\xi_{1,\mu\alpha}^\lambda \xi_1^\alpha + \xi_{1,\alpha}^\lambda \xi_{1,\mu}^\alpha). \end{aligned} \quad (3.21)$$

Using the method detailed in Refs. [10] and [11] we can extract from the general expressions Eqs. (3.20) and (3.21) the transformations behaviour of a particular metric function. We focus here on the curvature perturbation ψ , and find that it transforms at first order as [4, 5]

$$\widetilde{\psi}_1 = \psi_1 - \mathcal{H}\alpha_1. \quad (3.22)$$

At second order we get after some calculation ²

$$\widetilde{\psi}_2 = \psi_2 - \mathcal{H}\alpha_2 - \frac{1}{4} \mathcal{X}_2^k{}_k + \frac{1}{4} \nabla^{-2} \mathcal{X}_2^{ij}{}_{,ij}, \quad (3.24)$$

² The general expression given above in Eq. (3.21) gives the transformation for C_{2ij} , namely

$$2\widetilde{C}_{2ij} = 2C_{2ij} + 2\mathcal{H}\alpha_2\delta_{ij} + \xi_{2i,j} + \xi_{2j,i} + \mathcal{X}_{ij}, \quad (3.23)$$

and hence intermediate steps are necessary to extract the transformation of a particular component (see Refs. [10] and [11] for details).

where \mathcal{X}_{2ij} contains terms quadratic in the first order perturbations. For ease of presentation, on taking only scalar perturbations into account and working on large scales, where we can neglect gradients, \mathcal{X}_{2ij} takes the simple form

$$\mathcal{X}_{2ij} \equiv 2 \left[\left(\mathcal{H}^2 + \frac{a''}{a} \right) \alpha_1^2 + \mathcal{H} \alpha_1 \alpha'_1 \right] \delta_{ij} + 4 \alpha_1 (C'_{1ij} + 2 \mathcal{H} C_{1ij}) , \quad (3.25)$$

and Eq. (3.24) reduces to

$$\tilde{\psi}_2 = \psi_2 - \mathcal{H} \alpha_2 - \left(\mathcal{H}^2 + \frac{a''}{a} \right) \alpha_1^2 - \mathcal{H} \alpha_1 \alpha'_1 + 2 \alpha_1 (\psi'_1 - \mathcal{H} \psi_1) . \quad (3.26)$$

2. Third order

As above in the case of the transformation behaviour of a four-scalar at third order, the change under a gauge transformation of a two-tensor can be found applying the same methods as at second order. We therefore find that the metric tensor transforms at third order, from Eqs. (3.7) and (3.19), as

$$\begin{aligned} \widetilde{\delta g_{\mu\nu}^{(3)}} = & \delta g_{\mu\nu}^{(3)} + g_{\mu\nu,\lambda}^{(0)} \xi_3^\lambda + g_{\mu\lambda}^{(0)} \xi_{3,\nu}^\lambda + g_{\lambda\nu}^{(0)} \xi_{3,\mu}^\lambda \\ & + 3 \left[\delta g_{\mu\nu,\lambda}^{(1)} \xi_2^\lambda + \delta g_{\mu\lambda}^{(1)} \xi_{2,\nu}^\lambda + \delta g_{\lambda\nu}^{(1)} \xi_{2,\mu}^\lambda + \delta g_{\mu\nu,\lambda}^{(2)} \xi_1^\lambda + \delta g_{\mu\lambda}^{(2)} \xi_{1,\nu}^\lambda + \delta g_{\lambda\nu}^{(2)} \xi_{1,\mu}^\lambda \right. \\ & + \delta g_{\mu\nu,\lambda\alpha}^{(1)} \xi_1^\alpha + \delta g_{\mu\nu,\lambda}^{(1)} \xi_{1,\alpha}^\alpha + 2 \left[\delta g_{\mu\lambda,\alpha}^{(1)} \xi_1^\alpha \xi_{1,\nu}^\lambda + \delta g_{\lambda\nu,\alpha}^{(1)} \xi_1^\alpha \xi_{1,\mu}^\lambda + \delta g_{\lambda\alpha}^{(1)} \xi_{1,\mu}^\lambda \xi_{1,\nu}^\alpha \right] \\ & \left. + \delta g_{\mu\lambda}^{(1)} (\xi_{1,\nu\alpha}^\lambda \xi_1^\alpha + \xi_{1,\alpha}^\lambda \xi_{1,\nu}^\alpha) + \delta g_{\lambda\nu}^{(1)} (\xi_{1,\mu\alpha}^\lambda \xi_1^\alpha + \xi_{1,\alpha}^\lambda \xi_{1,\mu}^\alpha) \right] \\ & + \frac{3}{2} \left[2 g_{\mu\nu,\lambda\beta}^{(0)} \xi_2^\lambda \xi_1^\beta + g_{\mu\nu,\lambda}^{(0)} (\xi_{2,\beta}^\lambda \xi_1^\beta + \xi_{1,\beta}^\lambda \xi_2^\beta) + 2 g_{\lambda\beta}^{(0)} (\xi_{2,\mu}^\lambda \xi_{1,\nu}^\beta + \xi_{2,\nu}^\lambda \xi_{1,\mu}^\beta) \right. \\ & + (g_{\mu\lambda,\beta}^{(0)} \xi_{2,\nu}^\lambda + g_{\lambda\nu,\beta}^{(0)} \xi_{2,\mu}^\lambda + g_{\mu\lambda}^{(0)} \xi_{2,\beta\nu}^\lambda + g_{\lambda\nu}^{(0)} \xi_{2,\beta\mu}^\lambda) \xi_1^\beta + (g_{\beta\nu,\lambda}^{(0)} \xi_2^\lambda + g_{\lambda\nu}^{(0)} \xi_{2,\beta}^\lambda) \xi_{1,\nu}^\beta \\ & + (g_{\beta\nu,\lambda}^{(0)} \xi_2^\lambda + g_{\lambda\nu}^{(0)} \xi_{2,\beta}^\lambda) \xi_{1,\mu}^\beta + (g_{\mu\lambda,\beta}^{(0)} \xi_{1,\nu}^\lambda + g_{\lambda\nu,\beta}^{(0)} \xi_{1,\mu}^\lambda + g_{\mu\lambda}^{(0)} \xi_{1,\beta\nu}^\lambda + g_{\lambda\nu}^{(0)} \xi_{1,\beta\mu}^\lambda) \xi_2^\beta \\ & \left. + (g_{\beta\nu,\lambda}^{(0)} \xi_1^\lambda + g_{\lambda\nu}^{(0)} \xi_{1,\beta}^\lambda) \xi_{2,\nu}^\beta + (g_{\beta\nu,\lambda}^{(0)} \xi_1^\lambda + g_{\lambda\nu}^{(0)} \xi_{1,\beta}^\lambda) \xi_{2,\mu}^\beta \right] \\ & + \left\{ g_{\mu\nu,\alpha\beta\lambda}^{(0)} \xi_1^\alpha \xi_1^\beta \xi_1^\lambda + 3 g_{\mu\nu,\lambda\beta}^{(0)} \xi_1^\alpha \xi_{1,\alpha}^\beta + g_{\mu\nu,\lambda}^{(0)} \xi_{1,\alpha\beta}^\alpha \xi_1^\beta + g_{\mu\nu,\lambda}^{(0)} \xi_{1,\beta}^\alpha \xi_{1,\alpha}^\beta \xi_1^\lambda \right. \\ & + 3 g_{\mu\beta,\lambda}^{(0)} (\xi_1^\lambda \xi_{1,\alpha}^\beta \xi_{1,\nu}^\alpha + \xi_1^\alpha \xi_{1,\alpha}^\beta \xi_{1,\nu}^\lambda) + 3 g_{\mu\alpha,\lambda}^{(0)} \xi_1^\beta \xi_{1,\beta}^\alpha \xi_{1,\nu}^\lambda + 3 g_{\mu\alpha,\beta\lambda}^{(0)} \xi_1^\beta \xi_1^\alpha \xi_{1,\nu}^\lambda \\ & + g_{\mu\lambda}^{(0)} \left[\xi_1^\alpha \xi_1^\beta \xi_{1,\alpha\beta\nu}^\lambda + \xi_{1,\alpha}^\beta \xi_1^\lambda \xi_{1,\nu}^\alpha + \xi_1^\alpha \xi_{1,\alpha}^\beta \xi_{1,\beta\nu}^\lambda + 2 \xi_1^\beta \xi_{1,\alpha\beta}^\alpha \xi_{1,\nu}^\lambda + \xi_1^\beta \xi_{1,\alpha}^\lambda \xi_{1,\beta\nu}^\alpha \right] \\ & + 3 g_{\nu\beta,\lambda}^{(0)} (\xi_1^\lambda \xi_{1,\alpha}^\beta \xi_{1,\mu}^\alpha + \xi_1^\alpha \xi_{1,\alpha}^\beta \xi_{1,\mu}^\lambda) + 3 g_{\nu\alpha,\lambda}^{(0)} \xi_1^\beta \xi_{1,\beta}^\alpha \xi_{1,\mu}^\lambda + 3 g_{\nu\alpha,\beta\lambda}^{(0)} \xi_1^\beta \xi_1^\alpha \xi_{1,\mu}^\lambda \\ & + g_{\nu\lambda}^{(0)} \left[\xi_1^\alpha \xi_1^\beta \xi_{1,\alpha\beta\mu}^\lambda + \xi_{1,\alpha}^\beta \xi_1^\lambda \xi_{1,\mu}^\alpha + \xi_1^\alpha \xi_{1,\alpha}^\beta \xi_{1,\beta\mu}^\lambda + 2 \xi_1^\beta \xi_{1,\alpha\beta}^\alpha \xi_{1,\mu}^\lambda + \xi_1^\beta \xi_{1,\alpha}^\lambda \xi_{1,\beta\mu}^\alpha \right] \\ & \left. + 6 g_{\alpha\beta,\lambda}^{(0)} \xi_1^\lambda \xi_{1,\nu}^\beta + 3 g_{\alpha\lambda}^{(0)} \left[\xi_{1,\beta}^\lambda (\xi_{1,\mu}^\alpha \xi_{1,\nu}^\beta + \xi_{1,\nu}^\alpha \xi_{1,\mu}^\beta) + \xi_1^\beta (\xi_{1,\mu}^\alpha \xi_{1,\beta\nu}^\lambda + \xi_{1,\nu}^\alpha \xi_{1,\beta\mu}^\lambda) \right] \right\} . \quad (3.27) \end{aligned}$$

However, in this case it becomes even more obvious than in section II that the expressions at third order are of not inconsiderable size. This will also be clear from the Einstein tensor components and the evolution equations given below in Section V.

Now, following along the same lines as at second order, Eq. (3.27) gives the transformation for the spatial part of the metric at third order,

$$2\tilde{C}_{3ij} = 2C_{3ij} + 2\mathcal{H}\alpha_3\delta_{ij} + 2\xi_{3(i,j)} + \mathcal{X}_{3ij} , \quad (3.28)$$

where \mathcal{X}_{3ij} contains terms cubic in the first order perturbations. Extracting the curvature perturbation gives

$$\tilde{\psi}_3 = \psi_3 - \mathcal{H}\alpha_3 - \frac{1}{4}\mathcal{X}_3^k{}_k + \frac{1}{4}\nabla^{-2}\mathcal{X}_3^{ij}{}_{,ij}. \quad (3.29)$$

This expression is general, including scalar, vector, and tensor perturbations and is valid on all scales. However, as above we shall detail here only the expression valid for scalar perturbations and large scales, at third order an even bigger blessing than at second order, and find that \mathcal{X}_{3ij} takes then the simple form

$$\begin{aligned} \mathcal{X}_{3ij} \equiv & 2a^2\delta_{ij}\left\{-3\left[\alpha_2\psi'_1 + \frac{1}{2}\alpha_1\psi'_2 + \alpha_1\alpha'_1(\psi'_1 + 2\mathcal{H}\psi_1) + \alpha_1^2(\psi''_1 + 4\mathcal{H}\psi'_1) + 2\mathcal{H}\alpha_2\psi_1 + \mathcal{H}\alpha_1\psi_2 + 2\left(\frac{a''}{a} + \mathcal{H}^2\right)\alpha_1^2\psi_1\right]\right. \\ & \left.+ \left(\frac{a'''}{a} + 3\mathcal{H}\frac{a''}{a}\right)\alpha_1^3 + 3\left(\frac{a''}{a} + \mathcal{H}^2\right)\alpha_1^2\alpha'_1 + \mathcal{H}\alpha_1\left(\alpha_1''\alpha_1 + \alpha_1'^2\right) + \frac{3}{2}\left[\mathcal{H}(\alpha_1\alpha'_2 + \alpha_1'\alpha_2) + 2\left(\frac{a''}{a} + \mathcal{H}^2\right)\alpha_1\alpha_2\right]\right\}. \end{aligned} \quad (3.30)$$

Hence we finally get for the transformation of ψ_3

$$\begin{aligned} -\tilde{\psi}_3 = & -\psi_3 + \mathcal{H}\alpha_3 + \left(\frac{a'''}{a} + 3\mathcal{H}\frac{a''}{a}\right)\alpha_1^3 + 3\left(\frac{a''}{a} + \mathcal{H}^2\right)\alpha_1^2\alpha'_1 \\ & + \mathcal{H}\alpha_1\left(\alpha_1''\alpha_1 + \alpha_1'^2\right) + \frac{3}{2}\left[\mathcal{H}(\alpha_1\alpha'_2 + \alpha_1'\alpha_2) + 2\left(\frac{a''}{a} + \mathcal{H}^2\right)\alpha_1\alpha_2\right] \\ & - 3\left[\alpha_2\psi'_1 + \frac{1}{2}\alpha_1\psi'_2 + \alpha_1\alpha'_1(\psi'_1 + 2\mathcal{H}\psi_1) + \alpha_1^2(\psi''_1 + 4\mathcal{H}\psi'_1) + 2\mathcal{H}\alpha_2\psi_1 + \mathcal{H}\alpha_1\psi_2 + 2\left(\frac{a''}{a} + \mathcal{H}^2\right)\alpha_1^2\psi_1\right]. \end{aligned} \quad (3.31)$$

IV. GAUGE-INVARIANT QUANTITIES

In the previous section we have described how perturbations transform under a gauge shift. We can now use these results to construct gauge-invariant quantities, in particular the curvature perturbation on uniform density hypersurfaces, ζ . In this section, as before, we consider only scalar perturbations, and restrict ourselves to the large scale limit.

A. Defining hypersurfaces

From Eqs. (3.13), (3.15), and (3.18) we find the time slicing defining uniform density hypersurfaces at first, second and third order in the large scale limit as

$$\alpha_{1\delta\rho} = -\frac{\delta\rho_1}{\rho'_0}, \quad (4.1)$$

$$\alpha_{2\delta\rho} = -\frac{\delta\rho_2}{\rho'_0} + \frac{\delta\rho_1\delta\rho'_1}{\rho_0'^2}, \quad (4.2)$$

$$\alpha_{3\delta\rho} = -\frac{\delta\rho_3}{\rho'_0} + \frac{1}{2\rho_0'^2}\left[3(\delta\rho_1\delta\rho'_2 + \delta\rho'_1\delta\rho_2) - \frac{\delta\rho_1''\delta\rho_1^2}{\rho_0'} - 4\delta\rho_1'^2\frac{\delta\rho_1}{\rho_0'} + \rho_0''\delta\rho'_1\left(\frac{\delta\rho_1}{\rho_0'}\right)^2\right]. \quad (4.3)$$

Similarly, the temporal gauge transformation on uniform curvature hypersurfaces is defined by evaluating Eqs. (3.22), (3.26), and (3.31) and gives, at first, second and third order,

$$\alpha_{1\text{flat}} = \frac{\psi_1}{\mathcal{H}}, \quad (4.4)$$

$$\alpha_{2\text{flat}} = \frac{\psi_2}{\mathcal{H}} - \frac{4\psi_1^2}{\mathcal{H}} + \frac{\psi_1\psi'_1}{\mathcal{H}^2}, \quad (4.5)$$

$$\alpha_{3\text{flat}} = \frac{\psi_3}{\mathcal{H}} + \frac{1}{2\mathcal{H}^2}\left[3\psi_1'\psi_2 + \frac{\psi_1^2\psi''_1}{\mathcal{H}} + 6\mathcal{H}\psi_1\psi_2 + \frac{4\psi_1\psi_1^2}{\mathcal{H}}\right] - \frac{\psi_1^2\psi'_1}{\mathcal{H}^4}\left(\frac{a''}{a} - \frac{37}{2}\mathcal{H}^2\right) + \frac{8\psi_1^3}{\mathcal{H}}. \quad (4.6)$$

B. Constructing gauge-invariant quantities

We can now combine the results found so far to get gauge-invariant quantities, and as before choose the density perturbation on uniform curvature hypersurfaces and the curvature perturbation on uniform density slices as examples.

1. Energy density perturbation on uniform curvature hypersurfaces

A gauge invariant matter quantity of interest is the perturbation to the energy density on uniform curvature hypersurfaces. This is obtained by substituting the temporal gauge transformation components in the uniform density gauge (given in section IV A) into the appropriate transformation equation (that is, Eqs. (3.13), (3.15) or (3.18)). This gives, at first, second and third order, respectively,

$$\widetilde{\delta\rho_{1\text{flat}}} = \delta\rho_1 + \rho'_0 \frac{\psi_1}{\mathcal{H}}, \quad (4.7)$$

$$\widetilde{\delta\rho_{2\text{flat}}} = \delta\rho_2 + \rho'_0 \frac{\psi_2}{\mathcal{H}} + \frac{\psi_1^2}{\mathcal{H}} \left(\frac{\rho''_0}{\mathcal{H}} - 4\rho'_0 - \frac{\rho'_0}{\mathcal{H}^2} \left(\frac{a''}{a^2} - \mathcal{H}^2 \right) \right) + 2 \frac{\psi_1}{\mathcal{H}} \left(\rho'_0 \frac{\psi'_1}{\mathcal{H}} + \delta\rho_1 \right), \quad (4.8)$$

$$\begin{aligned} \widetilde{\delta\rho_{3\text{flat}}} = & \delta\rho_3 + \rho'_0 \frac{\psi_3}{\mathcal{H}} + \frac{3\rho'_0}{2\mathcal{H}^2} (2\psi_2\psi'_1 + \psi'_2\psi_1) + 3 \frac{\psi_1^2}{\mathcal{H}^3} \left[2(\rho'_0\psi_1 + \psi'_1\rho''_0) + \psi_1'' \right] + 3 \frac{\psi_2\psi_1}{\mathcal{H}^2} \left[\rho''_0 + 2\rho'_0\mathcal{H} - \rho'_0 \frac{a''}{a} \right] \\ & - 9\rho'_0 \frac{\psi_1^2\psi'_1}{\mathcal{H}^3} \left(\frac{a''}{a} - \mathcal{H} \right) + \frac{\psi_1^3}{\mathcal{H}^3} \left[\rho'''_0 - 3\rho''_0 \left(\frac{a''}{a} + 3\mathcal{H} \right) + 3\rho'_0\mathcal{H} \left(3\frac{a''}{a} - \mathcal{H} \right) + \rho'_0 \left(\left(\frac{a''}{a} \right)^2 - \frac{a'''}{a} \right) \right]. \end{aligned} \quad (4.9)$$

2. Curvature perturbation on uniform density hypersurfaces

The curvature perturbation on uniform density hypersurfaces, ζ , is defined as ³

$$-\zeta \equiv \widetilde{\psi_{\delta\rho}}. \quad (4.10)$$

This is obtained by substituting the temporal gauge transformation components in the uniform curvature gauge (given in section IV A) into the appropriate transformation equation (that is, Eqs. (3.22), (3.26) or (3.31)). Evaluating this on spatially flat hypersurfaces then gives, to first, second and third order, respectively,

$$\zeta_1 = -\mathcal{H} \frac{\delta\rho_1}{\rho'_0}, \quad (4.11)$$

$$\zeta_2 = -\frac{\mathcal{H}}{\rho'_0} \delta\rho_2 + 2\mathcal{H} \frac{\delta\rho_1 \delta\rho_1'}{\rho'^2_0} - \left[\mathcal{H} \frac{\rho''_0}{\rho'_0} - \left(\frac{a''}{a} + \mathcal{H}^2 \right) \right] \left(\frac{\delta\rho_1}{\rho'_0} \right)^2, \quad (4.12)$$

$$\begin{aligned} \zeta_3 = & -\mathcal{H} \frac{\delta\rho_3}{\rho'_0} + \frac{3\mathcal{H}}{\rho'^2_0} (\delta\rho_2' \delta\rho_1 + \delta\rho_1' \delta\rho_2) - \frac{3}{\rho'^2_0} \delta\rho_2 \delta\rho_1 \left[\frac{\mathcal{H}\rho''_0}{\rho'_0} - \left(\frac{a''}{a} + \mathcal{H}^2 \right) \right] - \frac{3\mathcal{H}}{\rho'^3_0} \delta\rho_1^2 \delta\rho_1'' - \frac{6\mathcal{H}}{\rho'^3_0} \delta\rho_1'^2 \delta\rho_1 \\ & - \frac{3}{\rho'^3_0} \delta\rho_1^2 \delta\rho_1' \left[2 \left(\frac{a''}{a} + \mathcal{H}^2 \right) - 3\mathcal{H} \frac{\rho''_0}{\rho'_0} \right] - \frac{\delta\rho_1^3}{\rho'^3_0} \left[3\mathcal{H} \left(\frac{\rho''_0}{\rho'_0} \right)^2 - \mathcal{H} \frac{\rho'''_0}{\rho'_0} + \frac{a'''}{a} + 3\mathcal{H} \frac{a''}{a} - 3 \frac{\rho''_0}{\rho'_0} \left(\frac{a''}{a} + \mathcal{H}^2 \right) \right]. \end{aligned} \quad (4.13)$$

There are different definitions of the curvature perturbation present in the literature, depending on different decompositions of the spatial part of the metric tensor. A different definition to the one above, as discussed in e.g. Ref. [11], was used by Maldacena in Ref. [21] to calculate the non-gaussianity from single field inflation, and was introduced by Salopek and Bond in Ref. [22]. They define the local scale factor $\tilde{a} \equiv e^\alpha$, then

$$e^{2\alpha} = a^2(\eta) e^{2\zeta} = a^2(\eta) (1 + 2\zeta_{\text{SB}} + 2\zeta_{\text{SB}}^2 + \frac{4}{3}\zeta_{\text{SB}}^3). \quad (4.14)$$

³ The sign is taken so as to coincide with ζ as defined in Ref. [20].

Comparing to the expansion from perturbation theory

$$e^{2\alpha} = a^2(\eta)(1 + 2\zeta), \quad (4.15)$$

one can obtain the relationship

$$\zeta = \zeta_{\text{SB}} + \zeta_{\text{SB}}^2 + \frac{2}{3}\zeta_{\text{SB}}^3. \quad (4.16)$$

Splitting this up order by order gives, at second order

$$\zeta_{2\text{SB}} = \zeta_2 - 2(\zeta_1)^2, \quad (4.17)$$

and, at third order,

$$\zeta_{3\text{SB}} = \zeta_3 - 6\zeta_2\zeta_1 + 8(\zeta_1)^3. \quad (4.18)$$

Note that it is this definition, (4.18), of the curvature perturbation which occurs in Ref. [16], though with different pre-factors since their perturbative expansion is defined differently⁴.

V. GOVERNING EQUATIONS

Having constructed gauge invariant quantities up to third order in the previous section, we now turn to the evolution and the field equations.

A. Fluid conservation equation

In this subsection, we give the energy momentum conservation equations for a fluid with non-zero pressure and in the presence of scalar, vector and tensor perturbations. The latter generalisation is important since at orders above linear order, all types of perturbation are coupled. It is also important to consider rotational fluids, since second order vorticity in the early universe is likely non-zero (see, for example, Ref. [18]). This agrees with Ref. [15], but is more general.

The Bianchi identities imply, through the Einstein field equations which relate the geometry of the space-time to its matter content, energy momentum conservation,

$$\nabla_\mu T^\mu{}_\nu = 0, \quad (5.1)$$

where ∇_μ here denotes the covariant derivative. Substituting the definition for the energy momentum tensor (2.7) into Eq. (5.1) gives energy conservation (the zero equation)

$$\begin{aligned} & \delta\rho' + 3\mathcal{H}(\delta\rho + \delta P) + (\rho_0 + P_0)(C'^i{}_i + v^i{}_{,i}) + (\delta\rho + \delta P)(C'^i{}_i + v^i{}_{,i}) + (\delta\rho + \delta P)_{,i}v^i \\ & + (\rho_0 + P_0)\left[(B^i + 2v^i)(v'_i + B'_i) + v^i{}_{,i}\phi - 2C'_{ij}C^{ij} + v^i(C^j{}_{j,i} + 2\phi_{,i}) + 4\mathcal{H}v^i(B_i + 2v_i)\right] \\ & + (\delta\rho + \delta P)\left[(B^i + 2v^i)(v'_i + B'_i) - 2C'_{ij}C^{ij} + v^i(C^j{}_{j,i} + 2\phi_{,i} + 4\mathcal{H}(v_i + B_i)) + v^i{}_{,i}\phi\right] \\ & + (\delta\rho' + \delta P')v^i(B_i + v_i) + (\delta\rho + \delta P)_{,i}\phi v^i + (\rho_0 + P_0)(2C^{ij}v_iv_j - B_iv^i\phi + v^iv_i\phi) \\ & - (\rho_0 + P_0)\left[2C^{ij}(C_{ij,k}v^k - 2v'_iv_j + B_iB'_j - 2C_i{}^kC'_{jk}) + \frac{1}{2}v^i{}_{,i}(\phi^2 - v^jv_j)\right] \\ & + (v^i + B^i)(2B'_i\phi - C^j{}_jv_i) - v^j\left\{B^iB_{i,j} + v^iv_{i,j} + \phi(v_j + 2v'_j + 3\mathcal{H}v_j - 2\phi_{,j} + C^i{}_{i,j})\right. \\ & \left. - 2\phi'B_j + 3C'_{ij}v^i\right\} + B^j(B^iC'_{ij} + B_j\phi' + v'_j\phi)\Big] = 0, \end{aligned} \quad (5.2)$$

⁴ It is worth mentioning that ζ_{SB} , the variable first introduced by Salopek and Bond and then employed for non-gaussianity calculations by Maldacena, is extremely Gaussian after slow-roll inflation, as opposed to other ζ variables which exhibit non-gaussianity, as can be seen e.g. from Eq. (4.17). We thank the anonymous referee for highlighting this issue.

and momentum conservation (the i -component)

$$\begin{aligned}
& \left[(\rho_0 + P_0)(v_i + B_i) \right]' + (\rho_0 + P_0)(\phi_{,i} + 4\mathcal{H}(v_i + B_i)) + \delta P_{,i} + \left[(\delta\rho + \delta P)(v_i + B_i) \right]' \\
& + (\delta\rho + \delta P)(\phi_{,i} + 4\mathcal{H}(v_i + B_i)) - (\rho'_0 + P'_0) \left[(2B_i + v_i)\phi - 2C_{ij}v^j \right] \\
& + (\rho_0 + P_0) \left[(v_i + B_i)(C^{j'}_{,j} + v^j_{,j}) - B_i(\phi' + 8\mathcal{H}\phi) + v^j(B_{i,j} - B_{j,i} + v_{i,j} + 8\mathcal{H}C_{ij}) \right. \\
& + (2C_{ij}v^j)' - \phi(v'_i + 2B'_i + 2\phi_{,i} + 4\mathcal{H}v_i) \left. \right] + (\rho'_0 + P'_0) \left[v^j(B_i v_j + B_j v_i + 2B_i B_j + \frac{1}{2}v_i v_j \right. \\
& - 2C_{ij}\phi) + \left(\frac{3}{2}v_i + 4B_i \right) \phi^2 \left. \right] + (\rho_0 + P_0) \left[\phi^2 \left(4B'_i + \frac{3}{2}v'_i + 2\mathcal{H}(8B_i + 3v_i) + 4\phi_{,i} \right) \right. \\
& + (v_i + 2B_i)(v'_j B^j - C^{j'}_{,j}\phi) + (v_i + B_i)(B'_j B^j - C^{j'}_{,j}C^{jk}) + v^j \left\{ 2B_i(B'_j + v'_j) \right. \\
& + B_j(2B'_i + 2\phi_{,i} + v'_i) + 2\mathcal{H}(v_i + 2B_i)(2B_j + v_j) + 2C_{ij}(C^{k'}_{,k} + v^{k,k} - 4\mathcal{H}\phi) \\
& + C^{k,k}_{,j}(B_i + v_i) + v_i(B'_j + \phi_{,j}) + v_j(B'_i + \phi_{,i}) + v^k(2C_{ik,j} - C_{jk,i}) \\
& \left. + (B_{j,i} - B_{i,j} - 2C'_{ij})\phi + 2C_{ik}v^{k,j} \right\} + \frac{1}{2}(v_i v_j v^j)' - 2C_{ij}v^{j'}\phi + B_i(4\phi'\phi - v_{j,i}v^j\phi) \left. \right] = 0. \tag{5.3}
\end{aligned}$$

Equation (5.2) highlights the coupling between tensor and scalar perturbations which occurs only at third order (and higher) in perturbation theory. At both linear and second order, no such coupling exists, since the only terms coupling the spatial metric perturbation, C_{ij} , to scalar perturbations contain either the trace or the divergence of C_{ij} and the tensor perturbation, h_{ij} is, by definition, transverse and trace-free. However, at third order, terms like $\delta\rho C'_{ij}C^{ij}$ occur in the energy conservation equation which, on splitting up order by order and decomposing C_{ij} becomes $\delta\rho_1 h'_{1ij} h_1{}^{ij}$. It is clear that this term only shows up at third order and beyond. Thus, as mentioned earlier, third order is the lowest order at which all the different types of perturbations couple to one another in the evolution equations, which will produce another physical signature of the full theory.

1. Scalars only

It will be useful to have energy and momentum conservation equations for only scalar perturbations. These equations are obtained by making the appropriate substitutions $C_{ij} = -\delta_{ij}\psi + E_{,ij}$, $v_i = v_{,i}$, and $B_i = B_{,i}$ into the above expressions. On doing so, we obtain the energy conservation equation

$$\begin{aligned}
& \delta\rho' + 3\mathcal{H}(\delta\rho + \delta P) + (\rho_0 + P_0)(\nabla^2 v + \nabla^2 E - 3\psi') + (\delta\rho + \delta P)(\nabla^2 v + \nabla^2 E - 3\psi') \\
& + (\delta\rho + \delta P)_{,i}v^{,i} + (\rho_0 + P_0) \left[(B^i + 2v^{,i})(v'_{,i} + B'_{,i}) + \nabla^2 v\phi - 2(\psi'(3\psi - \nabla^2 E) \right. \\
& - \psi\nabla^2 E' + E'_{,ij}E^{,ij}) + v_{,i}(2\phi_{,i} + \nabla^2 E_{,i} - 3\psi_{,i}) + 4\mathcal{H}v_{,i}(B_{,i} + 2v_{,i}) \left. \right] \\
& + (\delta\rho + \delta P) \left[(B_{,i} + 2v_{,i})(v'_{,i} + B'_{,i}) - 2(\psi'(3\psi - \nabla^2 E) - \psi\nabla^2 E' + E'_{,ij}E^{,ij}) \right. \\
& + v_{,i}(2\phi_{,i} - 3\psi_{,i} + \nabla^2 E + 4\mathcal{H}(v_{,i} + B_{,i})) + \nabla^2 v\phi \left. \right] + (\delta\rho' + \delta P')v_{,i}(B_{,i} + v_{,i}) \\
& + (\delta\rho + \delta P)_{,i}\phi v^{,i} + (\rho_0 + P_0)(v_{,i}v_{,i}\phi - B_{,i}v_{,i}\phi - 6\psi v_{,j}v_{,j} + E_{,ij}v_{,i}v_{,j}) \\
& - (\rho_0 + P_0) \left[2E_{,ij}(v^{,k}E_{,ijk} - 2v'_{,i}v_{,j} + B_{,i}B'_{,j} - 2E_{,i}{}^k E_{,kj}) - 2\nabla^2 E\psi_{,k}v^{,k} \right. \\
& - 2\psi(v^{,k}(\nabla^2 E_{,k} - 3\psi_{,k}) - 2v'_{,j}v_{,j} + B'_{,j}B_{,j} + 6\psi\psi') + \frac{1}{2}\nabla^2 v(\phi^2 - v_{,j}v_{,j}) \\
& + (v_{,i} + \nabla^2 E' + B_{,i})(2B'_{,i}\phi + 3\psi'v_{,i}) - v_{,j} \left\{ B_{,i}B_{,ij} + v_{,i}v_{,ij} - 2\phi'_{,j}B_{,j} - 3\psi'v_{,j} \right. \\
& + \phi(v_{,j} + 2v'_{,j} + 3\mathcal{H}v_{,j} - 2\phi_{,j} - 3\psi_{,j} + \nabla^2 E_{,j}) + E'_{,ij}v_{,i} \left. \right\} \\
& \left. + B_{,j}(B_{,j}\psi' + E'_{,ij}v_{,i} + B_{,j}\phi' + v'_{,j}\phi) \right] = 0, \tag{5.4}
\end{aligned}$$

and the momentum conservation equation

$$\begin{aligned}
& \left[(\rho_0 + P_0)(v_{,i} + B_{,i}) \right]' + (\rho_0 + P_0)(\phi_{,i} + 4\mathcal{H}(v_{,i} + B_{,i})) + \delta P_{,i} + \left[(\delta\rho + \delta P)(v_{,i} + B_{,i}) \right]' \\
& + (\delta\rho + \delta P)(\phi_{,i} + 4\mathcal{H}(v_{,i} + B_{,i})) - (\rho'_0 + P'_0) \left[(2B_{,i} + v_{,i})\phi + 2\psi v_{,i} - 2E_{,ij}v_{,j} \right] \\
& + (\rho_0 + P_0) \left[(v_{,i} + B_{,i})(\nabla^2 v - 3\psi' + \nabla^2 E') - B_{,i}(\phi' + 8\mathcal{H}\phi) + v_{,j} \left(v_{,ij} - 8\mathcal{H}(\psi\delta_{ij} - E_{,ij}) \right) \right. \\
& \left. - 2(\psi v_{,i} + E_{,ij}v_{,j})' - \phi(v'_{,i} + 2B'_{,i} + 2\phi_{,i} + 4\mathcal{H}v_{,i}) \right] + (\rho'_0 + P'_0) \left[v_{,j} (B_{,i}v_{,j} + B_{,j}v_{,i} + 2B_{,i}B_{,j} \right. \\
& \left. + \frac{1}{2}v_{,i}v_{,j} + 2\phi(\psi\delta_{ij} - E_{,ij})) + \left(\frac{3}{2}v_{,i} + 4B_{,i} \right) \phi^2 \right] + (\rho_0 + P_0) \left[(v_{,i} + 2B_{,i}) \left(v'_{,j}B_{,j} - (3\psi' - \nabla^2 E')\phi \right) \right. \\
& \left. + \phi^2 \left(4B'_{,i} + \frac{3}{2}v'_{,i} + 2\mathcal{H}(8B_{,i} + 3v_{,i}) + 4\phi_{,i} \right) + (v_{,i} + B_{,i}) \left(B'_{,j}B_{,j} - 3\psi'\psi + \psi'\nabla^2 E \right. \right. \\
& \left. \left. + \psi\nabla^2 E' - E_{,jk}E'_{,jk} \right) + v_{,j} \left\{ 2B_{,i}(B'_{,j} + v'_{,j}) + B_{,j}(2B'_{,i} + 2\phi_{,i} + v'_{,i}) + 2\mathcal{H}(v_{,i} + 2B_{,i})(2B_{,j} + v_{,j}) \right. \right. \\
& \left. \left. - (3\psi_{,j} - \nabla^2 E_{,j})(B_{,i} + v_{,i}) - 2(\psi\delta_{ij} - E_{,ij})(\nabla^2 v + \nabla^2 E' - 3\psi' - 4\mathcal{H}\phi) + v_{,i}(B'_{,j} + \phi_{,j}) \right. \right. \\
& \left. \left. + v_{,j}(B'_{,i} + \phi_{,i}) - 2\psi_{,j}v_{,i} + E_{,ikj}v_{,k} + \psi_{,i}v_{,j} + 2(\psi'\delta_{ij} - E_{,ij})\phi - 2\psi v_{,ij} + 2E_{,ik}v_{,k,j} \right\} \right. \\
& \left. + \frac{1}{2}(v_{,i}v_{,j}v_{,j})' + 2\psi v'_{,i}\phi - 2E_{,ij}v_{,j}'\phi + B_{,i}(4\phi'\phi - \nabla^2 v\phi) \right] = 0.
\end{aligned} \tag{5.5}$$

Note that, in the above, ∇^2 denotes the Laplacian; i.e. $\nabla^2 = \partial_j \partial^j$.

Considering the large scale limit, in which spatial gradients vanish, the energy conservation equation becomes

$$\begin{aligned}
& \delta\rho' + 3\mathcal{H}(\delta\rho + \delta P) - 3\psi'(\rho_0 + P_0) - 3\psi'(\delta\rho + \delta P) \\
& - 6\psi\psi'(\rho_0 + P_0) - 6\psi\psi'(\delta\rho + \delta P) + 12\psi^2\psi'(\rho_0 + P_0) = 0.
\end{aligned} \tag{5.6}$$

Splitting up perturbations order by order, this becomes

$$\begin{aligned}
& \delta\rho_3' + 3\mathcal{H}(\delta\rho_3 + \delta P_3) - 3\psi_3'(\rho_0 + P_0) - 9\psi_2'(\delta\rho_1 + \delta P_1) - 9\psi_1'(\delta\rho_2 + \delta P_2) \\
& - 18(\rho_0 + P_0)(\psi_2\psi_1' + \psi_1\psi_2') + 72\psi_1^2\psi_1'(\rho_0 + P_0) = 0.
\end{aligned} \tag{5.7}$$

In the uniform curvature gauge, where $\psi = 0$, this is

$$\delta\rho_{3\text{flat}}' + 3\mathcal{H}(\delta\rho_{3\text{flat}} + \delta P_{3\text{flat}}) = 0, \tag{5.8}$$

and in the uniform density gauge, where $\delta\rho = 0$,

$$3\mathcal{H}\delta P_{3\delta\rho} + 3\zeta_3'(\rho_0 + P_0) + 9\zeta_2'\delta P_{1\delta\rho} + 9\zeta_1'\delta P_{2\delta\rho} - 18(\rho_0 + P_0)(\zeta_2\zeta_1' + \zeta_1\zeta_2') - 72\zeta_1^2\zeta_1'(\rho_0 + P_0) = 0, \tag{5.9}$$

with ζ as defined in section IV B 2. This can be recast in the more familiar form by introducing the (gauge invariant) non-adiabatic pressure perturbation. At linear order the pressure perturbation can be expanded as

$$\delta P_1 = \frac{\partial P}{\partial S}\delta S_1 + \frac{\partial P}{\partial \rho}\delta\rho_1 \equiv \delta P_{\text{nad}1} + c_s^2\delta\rho_1. \tag{5.10}$$

This can be extended to second order [23] and higher by simply not truncating the Taylor series:

$$\delta P_{\text{nad}2} = \delta P_2 - c_s^2\delta\rho_2 - \frac{\partial c_s^2}{\partial \rho}\delta\rho_1^2, \tag{5.11}$$

$$\delta P_{\text{nad}3} = \delta P_3 - c_s^2\delta\rho_3 - 3\frac{\partial c_s^2}{\partial \rho}\delta\rho_2\delta\rho_1 - \frac{\partial^2 c_s^2}{\partial \rho^2}\delta\rho_1^3. \tag{5.12}$$

Thus, in the uniform density gauge, the pressure perturbation is equal to the non-adiabatic pressure perturbation at all orders. Then, Eq. (5.9) becomes

$$\zeta_3' + \frac{\mathcal{H}}{\rho_0 + P_0}\delta P_{\text{nad}3} = 6(\zeta_2\zeta_1' + \zeta_1\zeta_2') + 24\zeta_1^2\zeta_1' - \frac{3}{\rho_0 + P_0}(\zeta_2'\delta P_{\text{nad}1} + \zeta_1'\delta P_{\text{nad}2}). \tag{5.13}$$

In the case of a vanishing non-adiabatic pressure perturbation, ζ_1' and ζ_2' are zero and hence we see that ζ_3 is also conserved, on large scales. This was also found in Ref. [16], and previously in Ref. [24].

B. Klein Gordon equation

The energy momentum tensor for a canonical scalar field minimally coupled to gravity is easily obtained by treating the scalar field as a perfect fluid with energy-momentum tensor [25]

$$T^\mu{}_\nu = g^{\mu\lambda}\varphi_{,\lambda}\varphi_{,\nu} - \delta^\mu{}_\nu \left(\frac{1}{2}g^{\alpha\beta}\varphi_{,\alpha}\varphi_{,\beta} + U(\varphi) \right), \quad (5.14)$$

where the scalar field φ is split to third order as

$$\varphi(\eta, x^i) = \varphi_0(\eta) + \delta\varphi_1(\eta, x^i) + \frac{1}{2}\delta\varphi_2(\eta, x^i) + \frac{1}{3!}\delta\varphi_3(\eta, x^i), \quad (5.15)$$

and the potential U similarly as

$$U(\varphi) = U_0 + \delta U_1 + \frac{1}{2}\delta U_2 + \frac{1}{3!}\delta U_3, \quad (5.16)$$

where we define

$$\delta U_1 = U_{,\varphi}\delta\varphi_1, \quad \delta U_2 = U_{,\varphi\varphi}\delta\varphi_1^2 + U_{,\varphi}\delta\varphi_2, \quad \delta U_3 = U_{,\varphi\varphi\varphi}\delta\varphi_1^3 + 2U_{,\varphi\varphi}\delta\varphi_1\delta\varphi_2 + U_{,\varphi}\delta\varphi_3, \quad (5.17)$$

and making use of the shorthand notation $U_{,\varphi} \equiv \frac{\partial U}{\partial \varphi}$. Then, Eq. (5.1) gives the Klein Gordon equation

$$\begin{aligned} & \delta\varphi_3'' - \nabla^2\delta\varphi_3 + 4\mathcal{H}\delta\varphi_3' + \frac{\varphi_0''}{\varphi_0'}\delta\varphi_3' - \frac{3\delta\varphi_2''}{\varphi_0'}(2\varphi_0'\phi - \delta\varphi_1') - \frac{3}{\varphi_0'}(\nabla^2\delta\varphi_2\delta\varphi_1' + \delta\varphi_2'\nabla^2\delta\varphi_1) \\ & - \frac{6\delta\varphi_2'}{\varphi_0'}(\phi\varphi_0'' - 2\mathcal{H}\delta\varphi_1' + \varphi_0'\phi' + \varphi_0'B^i{}_{,i} - C^i{}_{,i}\varphi_0' + 4\mathcal{H}\varphi_0'\phi) - \frac{6(\delta\varphi_1')^2}{\varphi_0'}(\phi' + B^i{}_{,i} + 4\mathcal{H}\phi - C^i{}_{,i}) \\ & - \frac{6\delta\varphi_1'}{\varphi_0'}\left(2\phi\delta\varphi_1' - 4\varphi_0'\phi^2 + 2\varphi_0'\phi - \frac{1}{2}\delta\varphi_2' + B^iB_i\varphi_0'\right) - \frac{6\delta\varphi_1'}{\varphi_0'}\left[\varphi_0''(2\phi - 4\phi^2 + B^iB_i) \right. \\ & + 8\phi\varphi_0'(\mathcal{H} - 2\mathcal{H}\phi - 2\phi') + B^i(\varphi_0'B_i' + 2\delta\varphi_{1,i}' + 4\mathcal{H}B_i) + 2\varphi_0'\{B^i{}_{,i} + \phi' - C^i{}_{,i}(1 - 2\phi) \\ & - 2C^{ij}(B_{j,i} - C'_{ij}) + B^iC^j{}_{,j} - 2C^{ij}{}_{,j}B_i - B^i\phi_{,i} - 2B^i{}_{,i}\phi\} - 2C^{ij}\delta\varphi_{1,ij} \\ & + \delta\varphi_{1,i}(B^{i'} + 2\mathcal{H}B^i + 2C^{ij}{}_{,j} + \phi_{,i})\left. \right] - 3\delta\varphi_{2,i}(B^{i'} + C^j{}_{,j}{}^i + \phi_{,i} - 2C^{ij}{}_{,j} + 2\mathcal{H}B^i) \\ & - 6\delta\varphi_{1,i}\left[B^{i'} + C^j{}_{,j}{}^i - 2C^{ij}{}_{,j} + \phi_{,i} + \frac{C^j{}_{,j}{}^i}{\varphi_0'} + 2\mathcal{H}B^i - 2B_jC^{ij'} + B^iC^j{}_{,j}' + B^jB_{j,i} - B^jB^i{}_{,j} \right. \\ & - B^iB^j{}_{,j} - 2C^{ij}B_j' - 2C^{ij}C^k{}_{,k,j} + 4C^{ij}C_{jk}{}^k - 2C^{ij}\phi_{,j} - 2B^{i'}\phi - B^i\phi' - 2\phi_{,i}\phi - 4\mathcal{H}B_jC^{ij} \\ & - 4\mathcal{H}B^i\phi + 4C^{kj}C^i{}_{,k} - 2C^{kj}C_k{}^i{}_{,j}\left. \right] - 12B^i\left(\delta\varphi_{1,i}' + \frac{1}{2}\delta\varphi_{2,i}'\right) + 24\delta\varphi_{1,i}(B^i\phi + C^{ij}B_j) \\ & + 6\delta\varphi_{2,ij}C^{ij} - 6\delta\varphi_{1,ij}(4C^{kj}C_k{}^i - B^iB^j) - 24\mathcal{H}\varphi_0'\phi(1 - 2\phi + 4\phi^2 - \phi' + 3\phi'\phi) \\ & - 6\varphi_0''\left(2\phi(1 - 2\phi + 4\phi^2) + B^iB_i - 4B^iB_i\phi - 2B^iB^jC_{ij}\right) - 6C^{ij}{}_{,j}\varphi_0'(B_i\phi - 2B_i) \\ & + 6C^i{}_{,i}\varphi_0'(1 - 2\phi + 4\phi^2 - B^jB_j) - 6C^j{}_{,j,i}\varphi_0'(B^i - 2C^{ik}B_k - 2B^i\phi) - 6C^{ij'}\varphi_0'(2C_{ij} - B_iB_j \\ & - 4C_{ij}\phi - 4C_{kj}C^k{}_{,i}) - 12\varphi_0'C^{ij}\left[2B_iC_{jk}{}^k + 2B^kC_{ki,j} - B_{j,i} - B_iB_j' + B_i\phi_{,j} - B_kC_{ji}{}^k \right. \\ & + 2C_{ik}B^k{}_{,j} + 2B_{j,i}\phi - 4\mathcal{H}B_iB_j\left. \right] - 6\varphi_0'\left[\phi + B^i{}_{,i} + B^iB_i' - B^iB_iB^j{}_{,j} - B^iB_jB^j{}_{,i} - 4B^iB_i'\phi \right. \\ & + 2\mathcal{H}B^iB_i - 2B^iB_i\phi' - 8\mathcal{H}B^iB_i\phi + 4B^i\phi_{,i} - 2B^i{}_{,i}\phi + 4B^i{}_{,i}\phi^2 - B^i\phi_{,i}\left. \right] + 6U_{,\varphi}a^2 = 0. \end{aligned} \quad (5.18)$$

One can again see the coupling between first order tensor and scalar perturbations. For example, the $\delta\varphi_{1,i}C^{ij}\phi_{,j}$ contains a term that looks like $\delta\varphi_{1,i}h_1^{ij}\phi_{1,j}$, which occurs only at third order and beyond.

Again, we refrain from splitting up the perturbations order by order for ease of presentation. Once split up, one can then replace the metric perturbations by using the appropriate order field equations. We present the Einstein tensor at third order in the next section. Note also that Eq. (5.18) implicitly contains the Klein Gordon equations at first and second order. We refer the reader to, for example, Ref. [26], for a detailed exposition of the second order Klein Gordon equation.

C. Einstein tensor

The Einstein tensor, which describes the geometry of the universe, is defined as

$$G^\mu{}_\nu = R^\mu{}_\nu - \frac{1}{2}\delta^\mu{}_\nu R, \quad (5.19)$$

where $R^\mu{}_\nu$ is the Ricci curvature tensor and R is the Ricci scalar. Here, we give the components of the Einstein tensor up to third order:

$$\begin{aligned} a^2 G^0_0 &= -3\mathcal{H}^2 + \nabla^2 C^j{}_j - C_{ij},{}^{ij} + 2\mathcal{H}(-C^{i'}{}_i + B^i{}_{,i} + 3\mathcal{H}\phi) + C^j{}_{j,i}(\frac{1}{2}C^k{}_{k,i} - 2C^{ik}{}_{,k}) + C'_{ij}(\frac{1}{2}C^{ij'} - B^j{}_{,i}) \\ &\quad + B^i\left[C^{j'}{}_{j,i} - C'_{ij},{}^j + \frac{1}{2}(\nabla^2 B_i - B_{j,i}{}^j) + 2\mathcal{H}(C^j{}_{j,i} - 2C_{ij},{}^j - \phi_{,i})\right] + 2C^{ij}\left[2C_{jk,i}{}^k - C^k{}_{k,ij} - \nabla^2 C_{ij}\right. \\ &\quad \left.+ 2\mathcal{H}(C'_{ij} - B_{i,j})\right] + C_{jk,i}(C^{ik},{}^j - \frac{3}{2}C^{jk},{}^i) + C^{i'}{}_i(B_{j,i}{}^j - \frac{1}{2}C^{j'}{}_j + 4\mathcal{H}\phi) + 2C^{ij},{}_i C_{jk},{}^k \\ &\quad + \frac{1}{4}B_{j,i}(B^i{}_{,j} + B^j{}_{,i}) - 3\mathcal{H}^2(4\phi^2 - B_i B^i) - \frac{1}{2}B^i{}_{,i}B_{j,i}{}^j - 4\mathcal{H}B^i{}_{,i}\phi + \mathbb{G}^0_0, \end{aligned} \quad (5.20)$$

$$\begin{aligned} a^2 G^0_i &= C^k{}_{k,i} - C'_{ik},{}^k - \frac{1}{2}(B_{k,i}{}^k - \nabla^2 B_i) - 2\mathcal{H}\phi_{,i} + 8\mathcal{H}\phi_{,i}\phi + C'_{ij}(2C^{kj}{}_{,k} - C^k{}_{k,j} + \phi_{,j}) - C^{j'}{}_j\phi_{,i} \\ &\quad + 2C^{kj}\left[C'_{ik,j} - C'_{jk,i} + \frac{1}{2}(B_{k,ij} - B_{i,kj})\right] + B^j(C_{kj,i}{}^k - C^k{}_{k,ij} + C_{ik,k}{}^j - \nabla^2 C_{ij} - 2\mathcal{H}B_{j,i}) \\ &\quad - \frac{1}{2}(B_{i,j} + B_{j,i})\phi_{,j} + (B_{i,j} - B_{j,i})\left(\frac{1}{2}C^k{}_{k,j} - C^{jk}{}_{,k}\right) - C_{ik,j}(B^k{}_{,j} - B^j{}_{,k}) + B^j{}_{,j}\phi_{,i} \\ &\quad + \phi\left[B_{j,i}{}^j - \nabla^2 B_i + 2(C'_{ij},{}^j - C^{j'}{}_j,{}_i)\right] - C^{kj'}C_{kj,i} + \mathbb{G}^0_i, \end{aligned} \quad (5.21)$$

$$\begin{aligned} a^2 G^i_j &= C^{i''}{}_j + 2\mathcal{H}C^{i'}{}_j - \frac{1}{2}(B^{i'}{}_{,j} + B_{j,i}{}^{i'}) - C^l{}_{l,j}{}^i + C^i{}_{l,j}{}^l - \phi_{,j}{}^i - \nabla^2 C^i{}_j + C_{jl},{}^{il} - \mathcal{H}(B^i{}_{,j} + B_{j,i}{}^i) \\ &\quad + \delta^i{}_j\left\{\left(\mathcal{H}^2 - \frac{2a''}{a}\right)(1 - 2\phi) + 2\mathcal{H}(B^k{}_{,k} - C^{k'}{}_k + \phi') + B^{k'}{}_k - C^{kl}{}_{,kl} - C^{k''}{}_k + \nabla^2(\phi + C^l{}_l)\right\} \\ &\quad + B^k\left[C_{jk},{}^{i'} + C^{i'}{}_{k,j} - 2C^{i'}{}_{j,k} + 2\mathcal{H}(C_{jk},{}^i + C^i{}_{k,j} - C^i{}_{j,k}) + \frac{1}{2}(B_{j,i}{}^k + B^i{}_{,jk} - 2B_{k,i}{}^j)\right] \\ &\quad + (C^k{}'_k - \phi' - B^k{}_{,k})(C^{i'}{}_j - \frac{1}{2}(B^i{}_{,j} + B_{j,i}{}^i)) + C^{ik'}(B_{j,k} - 2C'_{kj}) + C'_{kj}B^i{}_{,k} + \phi_{,i}\phi_{,j} \\ &\quad + (B^{k'} - 2C^{kl}{}_{,l} + C^l{}_{l,k} + \phi_{,k})(C_{jk},{}^i + C^i{}_{k,j} - C^i{}_{j,k}) + \frac{1}{2}B^i(B_{k,j}{}^k - \nabla^2 B_j + 4\mathcal{H}\phi_{,j} - 2C^{k'}{}_{k,j} + 2C'_{kj},{}^k) \\ &\quad + 2C^{ik}\left[\frac{1}{2}(B'_{j,k} + B'_{k,j}) - C''_{kj} + \phi_{,jk} - C_{kl,j}{}^l - C_{jl,k}{}^l + \nabla^2 C_{kj} + C^l{}_{l,jk} + \mathcal{H}(B_{j,k} + B_{k,j} - 2C'_{kj})\right] \\ &\quad - \frac{1}{2}(B_{k,i}{}^k B^k{}_{,j} + B_{j,i}{}^k B^i{}_{,k}) + \phi\left[(B_{j,i}{}^{i'} + B^{i'}{}_{j,i} + 2\phi_{,j}{}^i + 2\mathcal{H}(B_{j,i}{}^i + B^i{}_{,j}) - 2C^{i''}{}_j - 4\mathcal{H}C^{i'}{}_j)\right] \\ &\quad + 2(C^i{}_{k,l}C^k{}_{j,l} - C^l{}_{j,i}C^i{}_{k,l} + C^{kl}{}_{,j}C_{kl},{}^i) + 2C^{kl}\left[C_{kl,j}{}^i - C_{jl},{}^i{}_k - C^i{}_{l,jk} + C^i{}_{j,kl}\right] + \mathbb{G}^i_{oj} \\ &\quad + \delta^i{}_j\left\{\left(\mathcal{H}^2 - \frac{2a''}{a}\right)(4\phi^2 - B_k B^k) + 2\phi\left[C^{k''}{}_k - B^{k'}{}_k - \nabla^2\phi + 2\mathcal{H}(C^{k'}{}_k - 2\phi' - B^k{}_{,k})\right]\right. \\ &\quad \left.+ B^k\left[2C^{l'}{}_{l,k} - 2C'_{kl},{}^l + \nabla^2 B_k - B_{l,k}{}^l + 2\mathcal{H}(B'_k - \phi_{,k} - 2C^l{}_{k,l} + C^l{}_{l,k})\right] + C^{kl'}\left(\frac{3}{2}C'_{kl} - B_{l,k}\right)\right. \\ &\quad \left.+ 2C^{kl}\left[C''_{kl} - \nabla^2 C_{kl} + 2\mathcal{H}C'_{kl} + 2C_{lm,k}{}^m - C^m{}_{m,kl} - 2\mathcal{H}B_{l,k} - B'_{l,k} - \phi_{,kl}\right] + 2B^{k'}(C^l{}_{l,k} - C_{kl},{}^l)\right. \\ &\quad \left.+ C^{k'}{}_k\left(B^l{}_{,l} - \frac{1}{2}C^{l'}{}_l\right) + 2C^{kl},{}_k C_{lm},{}^m + C_{lm,k}\left(C^{km},{}^l - \frac{3}{2}C^{lm},{}^k\right) - C^l{}_{l,k}\left(2C^k{}_{m,m} - \frac{1}{2}C^m{}_{m,k}\right)\right. \\ &\quad \left.+ \phi'\left(C^{k'}{}_k - B^k{}_{,k}\right) - \frac{1}{4}(2B^k{}_{,k}B_{l,i}{}^l - B_{l,k}B^k{}_{,l} - 3B^l{}_{,k}B_{l,i}{}^k) + \phi_{,k}(C^l{}_{l,k} - 2C^{lk}{}_{,l} - \phi_{,k}) + \mathbb{G}^i_{dj}\right\}, \end{aligned} \quad (5.22)$$

where $\mathbb{G}^0_0, \mathbb{G}^0_i, \mathbb{G}^i_j$ are the third order corrections (the latter split into a diagonal part \mathbb{G}^i_{oj} , and an off diagonal part \mathbb{G}^i_{dj}) which we give in the appendix as Eqs. (A1), (A2), (A3) and (A4), respectively. Note that, in calculating the third order components given above, we have implicitly obtained the full second order Einstein tensor components for fully general perturbations (i.e. including all scalar, vector and tensor perturbations). We shall extend this second order analysis, by presenting all the geometric and matter tensors as well as conservation and constraint equations in full generality, in a future publication [27].

VI. DISCUSSION AND CONCLUSIONS

In this paper we have developed the essential tools for cosmological perturbation theory at third order. Starting with the definition of the active gauge transformation we have extended the work of Ref. [11] to third order, and derived gauge invariant variables, namely the curvature perturbation on uniform density hypersurfaces, ζ_3 , and the density perturbation on uniform curvature hypersurfaces. We also relate the curvature perturbation ζ_3 , obtained using the spatial metric split of Ref. [11] to that introduced by Salopek and Bond [22], which is also popular at higher order.

We have then calculated the energy and momentum conservation equations for a general perfect fluid at third order, including all scalar, vector and tensor perturbations. The Klein Gordon equation for a canonical scalar field minimally coupled to gravity is also presented. We highlight the coupling in these conservation equations between scalar and tensor perturbations which only occurs at third order and above. Finally, we have presented the Einstein tensor components to third order. No large scale approximation is employed for the tensor components or the conservation equations. All equations are given without specifying a particular gauge, and can therefore immediately be rewritten in whatever choice of gauge is desired. However, as examples to illustrate possible gauge choices, we give the energy conservation equation on large scales (and only allowing for scalar perturbations) in the flat and the uniform density gauge. This gives an evolution equation for the curvature perturbation ζ_3 , Eq. (5.13). As might be expected from fully non-linear calculations [28] and second order perturbative calculations [29], the curvature perturbation is also conserved at third order on large scales in the adiabatic case. It is worth noting that higher order perturbation theory, as discussed in this paper, has the advantage of being valid on all scales whereas fully non-linear methods, such as the δN formalism, are only valid in the large scale limit.

Another application of our third order variables and equations, in particular the Klein Gordon equation (5.18), is the calculation of the trispectrum by means of the field equations. Whereas calculations of the trispectrum so far derive the trispectrum from the fourth order action, it should also be possible to use the third order field equations instead. The equivalence of the two approaches for calculating the bispectrum, using the third order action or the second order field equations, has been shown in Ref. [30]. Having included tensor as well as scalar perturbations it will be in particular interesting to see and be an important consistency check for the theory whether we arrive at the same result as Ref. [31].

A final advantage of extending perturbation theory to third order is that, in doing so, one obtains a deeper insight into the second order theory. Also second order perturbation theory, despite remaining challenging, becomes less daunting having explored some of the third order theory.

Acknowledgments

The authors are grateful to Jim Lidsey, David Lyth, Misao Sasaki and David Wands for useful comments and discussions. AJC is supported by the Science and Technology Facilities Council (STFC). We used the computer algebra package CADABRA [32] to assist with the calculations, and thank Kasper Peeters for useful discussions on using the package.

APPENDIX A: EINSTEIN TENSOR

Here, we give the third order corrections to the components of the Einstein tensor. We do not split up perturbations order by order.

$$\begin{aligned}
\mathbb{G}^0_0 = & 2C^{ij} \left[2C_{ik}(C^l_{l,j}{}^k - 2C^{kl}{}_{,jl} + \nabla^2 C_j{}^k - 2\mathcal{H}C_j'^k) + (2C_{jk,i} - C_{ij,k})(C^l_{l,j}{}^k - 2C^k{}_{l,j}{}^l) + C_{il,k}(3C_j{}^l{}^k - C_j{}^k{}_{,l}) \right. \\
& + C^k{}_{k,i}(2C^j_{l,j}{}^l - \frac{1}{2}C^l_{l,j}) + B^i \left\{ C_j'^k{}^k - C^{k'}{}_{k,j} + \frac{1}{2}(B_{k,j}{}^k - \nabla^2 B_j) + \mathcal{H}(4C_{jk}{}^k + 2\phi_{,j} - 3\mathcal{H}B_j - 2C^k{}_{k,j}) \right\} \\
& + C'_{ik}(B^k{}_{,j} + B_j{}^k) - 2C_{ik}{}^k C_{jl}{}^l - (B^k{}_{,k} + \phi)(C'_{ij} - B_{j,i}) - \frac{1}{4}B_{k,i}(B^k{}_{,j} + 2B_j{}^k) - \frac{1}{4}B_{i,k}B_j{}^k - C^{k'}{}_{k,i}B_{j,i} \\
& + 8\mathcal{H}C_{ik}B^k{}_{,j} \left. \right] + 2C^{kj}B^i \left[C'_{ik,j} - C'_{jk,i} + \frac{1}{2}(B_{k,ij} - B_{i,jk}) + 2\mathcal{H}(2C_{ik,j} - C_{kj,i}) \right] + C^{ij'} \left[2B^k(C_{ki,j} - C_{ji,k}) \right. \\
& + B_i(2C_{jk}{}^k - C^k{}_{k,j}) + B_i\phi_{,j} + (2B_{j,i} - C'_{ij})\phi \left. \right] + 8\mathcal{H}\phi^2(B^i{}_{,i} - C^{i'}{}_{i,i}) + (B_{i,j} - B_{j,i})(\frac{1}{2}B^i C^k{}_{k,j} - B^i C^{jk}{}_{,k}) \\
& - \frac{1}{2}(B_{i,j} + B_{j,i})(\phi B^j{}_{,i} + B^i\phi_{,j}) - 2\mathcal{H}B^i B^j B_{j,i} + B^j{}_{,j} \left[B^i(2C_{ik}{}^k - C^k{}_{k,i} + \phi_{,i} - 2\mathcal{H}B_i) + B^i_i\phi - 2C^{i'}{}_{i,i}\phi \right] \\
& + B_{k,j}B^i(C^{kj}{}_{,i} - 2C^k{}_{k,j}) + 2\mathcal{H}B^i \left[C_j'^j B_i + 2\phi(2C_{ij}{}^j - C^j{}_{j,i} + 2\phi_{,i} - 3\mathcal{H}B_i) \right] + C^{i'}{}_i C_j'^j \phi \\
& + B^i B^j (2C_{jk,i}{}^k - C^k{}_{k,ij} - \nabla^2 C_{ij}) + B^i C_j'^j (C^k{}_{k,i} - 2C_{ik}{}^k - \phi_{,i}) + B^i\phi(B^j{}_{,ij} + 2C'_{ij}{}^j - 2C^j{}_{j,i} - \nabla^2 B_i), \quad (A1)
\end{aligned}$$

$$\begin{aligned}
\mathbb{G}^0_i = & C_{kj} \left[2C^{jl} (C^k{}_{l,i}{}^j - C_{il}{}^{k'}) + C_i{}^j (C^l_{l,j}{}^k - 2C^{kl}{}_{,l} - 2\phi_{,k}) + C'_{il} (C^{jk}{}_{,l} - 2C^{kl}{}_{,j}) + 2C^{jl'} C^k{}_{l,i} \right] \\
& + C^{kj} \left[(B_{j,i} - B_{i,j})(C^l_{l,k} - C_{kl}{}^l) + (B_{l,i} - B_{i,l})(C_{jk}{}^l - 2C^l{}_{k,j}) + 2(B_{j,l} - B_{l,j})(C^l_{ik} - C_i{}^l{}_{,k}) + (B_{j,i} + B_{i,j})\phi_{,k} \right. \\
& + 2(C'_{jk} - B_{k,j})\phi_{,i} + 2\phi(2C'_{jk,i} - 2C'_{ik,j} + B_{i,kj} - B_{k,ij}) \left. \right] + 2B_j \left[C^{kj} (C^l_{l,ik} - C_{kl,i}{}^l - C_{il,k}{}^l + \nabla^2 C_{ik}) \right. \\
& + C^{kl} (C_{kl,i}{}^j - C^j{}_{l,ik} - C_{il,k}{}^j + C_i{}^j{}_{,kl}) + (C^l_{l,k} - C^{kl}{}_{,l})(C^j{}_{k,i} + C_{ik}{}^j - C^j{}_{i,k}) + C_{il,k}(C^{jl}{}_{,k} - C^{jk}{}_{,l}) \\
& + \frac{1}{2}C_{kl,i}C^{kl}{}_{,j} \left. \right] + 2C'_{ij}\phi(C^k{}_{k,j} - 2C^{kj}{}_{,k}) + B^j \left[B_j(C'_{ik}{}^k - C^{k'}{}_{k,i} + \frac{1}{2}(B_{k,i}{}^k - \nabla^2 B_i)) + C'_{ik}B_j{}^k \right. \\
& + B_{j,i}(B^k{}_{,k} - C^{k'}{}_{k,i}) + \frac{1}{2}B_{j,k}(2C_i'^k - B^k{}_{,i}B_i{}^k) \left. \right] + 2\phi \left[(B_{i,j} - B_{j,i})(C^{kj}{}_{,k} - \frac{1}{2}C^k{}_{k,j}) + \phi_{,j}(B_{i,j} + B_{j,i}) \right. \\
& + C'_{kj}C^{kj}{}_{,i} + 2C_j'^j\phi_{,i} - 2C'_{ij}\phi_{,j} - B_{k,j}(C_i{}^j{}^k - C_i{}^k{}_{,j}) - 2B_{j,i}\phi_{,k} \left. \right] + 2C_{kj}C^{jl}(B_{i,l}{}^k - B_{l,i}{}^k) \\
& + B^j \left[\phi_{,i}(C^k{}_{k,j} - 2C_{jk}{}^k + 4\mathcal{H}B_j) + 2\phi(C^k{}_{k,ij} - C_{jk}{}^k{}_i - C_{ik,j}{}^k + \nabla^2 C_{ij} + 4\mathcal{H}B_{j,i}) + 2\mathcal{H}(B^k C_{jk,i} + 2C_{kj}B^k{}_{,i}) \right. \\
& + \phi_{,k}(C_{ij,k} - C_{ik,j} + C_{jk,i}) \left. \right] + 2\phi^2(2C_j'^j{}_{,i} - 2C'_{ij}{}^j - B_{j,i}{}^j + \nabla^2 B_i - 12\mathcal{H}\phi_{,i}), \quad (A2)
\end{aligned}$$

$$\begin{aligned}
\mathbb{G}^i_{0j} = & 2C^{ik}C_{kl} \left[2C^{l''}{}_j - B^{l'}{}_{,j} - B_j{}^{l'} - 2C^m{}_{m,j}{}^l + 2C^l{}_{m,j}{}^m + 2C_{jm}{}^{lm} - 2\nabla^2 C^l{}_j - 2\phi_{,j}{}^l \right] \\
& + 4C^{ik}C^{lm}(C_{km,jl} - C_{lm,kj} + C_{jm,kl} - C_{jk,lm}) + 4C^{km}C_{kl}(C_{jm,i}{}^l - C^l{}_{m,j}{}^i + C^i{}_{m,j}{}^l - C^i{}_{j,m}{}^l) \\
& + C^{ik} \left[2(2C^{lm}{}_{,m} - C^m{}_{m,l} - \phi_{,l} - B^{l'}{}_{,l}) (C_{kl,j} + C_{jl,k} - C_{jk,l}) + 4C_{jm,l}(C^l{}_{k,m} - C^m{}_{k,l}) - 2C_{lm,j}C^{lm}{}_{,k} \right. \\
& - 2C'_{jk}(C^{l''}{}_l - B^l{}_{,l} - \phi') + 2C'_{jl}(2C^{l''}{}_k - B_k{}^l) + C^{l'}{}_l(B_{k,j} + B_{j,k}) - 2C'_{kl}B_j{}^l - (\phi' + B^l{}_{,l})(B_{k,j} + B_{j,k}) \\
& + B_{l,j}B^l{}_{,k} + B_{j,l}B_k{}^l - 2\phi_{,k}\phi_{,j} - 2\phi(B'_{k,j} + B'_{j,k} + \phi_{,jk} + 4\mathcal{H}C_{kl}(2C^{l'}{}_j - B^l{}_{,j} - B_j{}^l) - 4\mathcal{H}B_k\phi_{,j}) \left. \right] \\
& + C^{kl} \left[2(C^m{}_{m,l} - 2C_{lm}{}^m)(C^i{}_{j,k} - C^i{}_{k,j} - C_{jk}{}^i) + 2(2C^m{}_{l,k} - C_{kl}{}^m)(C_{jm,i}{}^i + C^i{}_{m,j} - C^i{}_{j,m}) - 4C_{km,i}C^m{}_{l,j} \right. \\
& + 4(C_{jl,m} - C_{jm,l})(C^{im}{}_{,k} - C^i{}_{k,m}) - 2C^{i'}{}_j C'_{kl} + 4C^{i'}{}_k C'_{jl} + (B^i{}_{,j} + B_j{}^i)(C'_{kl} - B_{l,k}) - 2\phi_{,l}(C_{jk,i}{}^i + C^i{}_{k,j} - C^i{}_{j,k}) \\
& + 2C^{i'}{}_j B_{l,k} - 2C^{i'}{}_k B_{j,l} - 2C'_{jk}B^i{}_{,l} + B_{k,i}B_{l,j} - 2B'_k(C_{jl,i}{}^i + C^i{}_{l,j} - C^i{}_{j,l}) \left. \right] - B^k B_k \left[C^{i''}{}_j - \frac{1}{2}(B^{i'}{}_{,j} + B_j{}^{i'}) \right. \\
& - \phi_{,j} - \mathcal{H}(B^i{}_{,j} + B_j{}^i - 2C^{i'}{}_j) \left. \right] + B^i B^{k'} \left[C^l{}_{l,jk} - C^l{}_{k,jl} - C^l{}_{j,kl} + \nabla^2 C_{jk} + 2\mathcal{H}B_{k,j} \right] \\
& + B^k B^l \left[C_{kl,i}{}^j - C_{jl,i}{}^k - C^i{}_{l,jk} + C^i{}_{j,kl} \right] + B^i \left[C'_{jk}(C^l_{l,k} - 2C^{kl}{}_{,l} - \phi_{,k}) + C'_{kl}C^{kl}{}_{,j} + \frac{1}{2}\phi_{,k}(B_{k,j} + B_{j,k}) \right]
\end{aligned}$$

$$\begin{aligned}
& + \left(\frac{1}{2} C_{l,k}^{l,k} - C_{l,l}^{kl} \right) (B_{k,j} - B_{j,k}) + B_{l,k} (C_{j,l}^{l,k} - C_{j,l}^{k,l}) - \phi_{,j} (B_{,k}^k - C_{,k}^{k'}) \Big] + 4\mathcal{H}\phi^2 (2C_{,j}^{i'} - B_{,j}^i - B_{j,i}) \\
& + B^k \Big[C_{,k}^i (2C_{,j}^{l'} - 2C_{j,l}^{l'} - B_{,j}^l + \nabla^2 B_j) + C^{il} (4C_{j,l,k}' - 2C_{kl,j}' - C_{j,k,l}' + 2B_{k,j,l} - B_{l,k,j} - B_{j,kl}) \\
& + C_{,k}^l (4C_{j,l}^{i'} - 2C_{,j}^{i'} - C_{j,l}^{i'} + 2B_{l,j}^i - B_{j,l}^i - B_{,j}^i) + C_{,j}^{i'} (2C_{kl,l} - B_k' - C_{l,k}^l + \phi_{,k}) + 2C^{il'} (C_{j,l,k} - C_{j,k,l}) \\
& + 2C_{j,l}' (C_{,k}^{il} - C_{,k}^{i,l}) + \frac{1}{2} (B_k' + C_{l,k}^l - 2C_{kl,l}') (B_{j,i} + B_{,j}^i) + (C_{,j}^{l'} + C_{,j}^{il} - C_{,j}^{i,l}) (B_{k,l} - B_{l,k} - 2C_{kl}' - 4\mathcal{H}C_{kl}) \\
& + (C_{j,k}^i + C_{,k}^{i,j} - C_{,j,k}^i) (C_{,l}^{l'} - \phi' - B_{,l}^l) + (B_{,l}^i - 4\mathcal{H}C^{il}) (C_{j,k,l} - C_{j,l,k}) + (B_{,l}^i - 4\mathcal{H}C^{il}) C_{kl,j} + B_{,j}^l C_{kl,i} \\
& - \frac{1}{2} \phi_{,k} (B_{,j}^i + B_{j,i}) + B_{j,l} (C_{,k}^{i,l} - C_{,k}^{il}) + B_{k,i} \phi_{,j} + B_{k,j} \phi_{,i} \Big] + 2\phi \Big[B^i (C_{,k}^{k'} - C_{j,k}^{k'} + \frac{1}{2} (\nabla^2 B_j - B_{,k}^k)) \\
& + B^k (2C_{,j,k}^{i'} - C_{,j,k}^{i'} - C_{j,k}^{i'} + B_{k,j}^i - \frac{1}{2} (B_{j,k}^i + B_{,j,k}^i)) \Big] + (B^{k'} + \phi_{,k} + 4\mathcal{H}B^k) (C_{,j,k}^i - C_{,k,j}^i - C_{j,k}^i) \\
& + C_{,k}^{i'} (2C_{,j}^{k'} - B_{j,k}^k) + 2C^{ik} C_{j,k}'' - C_{j,k}' B_{,k}^i + \frac{1}{2} (B_{,j}^i + B_{j,i}) (C_{,k}^{k'} - 2\phi' - B_{,k}^k) + C_{,j}^{i'} (B_{,k}^k - C_{,k}^{k'} + 2\phi') \\
& + 4\phi_{,j} (2\mathcal{H}B^i - \phi_{,i}) - 4\mathcal{H}C^{ik} (2C_{k,j}' + B_{k,j} + B_{j,k}) + \frac{1}{2} (B_{k,i} B_{,j}^k + B_{,k}^i B_{j,k}) \Big], \tag{A3}
\end{aligned}$$

$$\begin{aligned}
\mathbb{G}_{dj}^i & = 4C^{kl} C_{km} \Big[\nabla^2 C_l^m - C^{m''}{}_l + C^n{}_{n,l}{}^m - 2C^{mn}{}_{,nl} + B^{m'}{}_l + \phi_{,l}{}^m + 2\mathcal{H}(B^m{}_{,l} - C^{m'}{}_l) \Big] \\
& + C^{kl} \Big[4C_{mn} (C^{mn}{}_{,kl} - 2C_l^n{}_{,k}{}^m) + 2C_{km}' (B^m{}_{,l} + B_l{}^m - 3C^{m'}{}_l) + 2(C^n{}_{n,l}{}^m - 2C^{mn}{}_{,n}) (2C_{lm,k} - C_{kl,m}) \\
& + B^{m'} (4C_{ml,k} - C_{kl,m}) + 2B_k' (2C_{lm}{}^m - C^m{}_{m,l}) + C^m{}_{m,k} (4C_{ln}{}^n - C^m{}_{n,l}) + C_{mn,k} (3C^{mn}{}_{,l} - 4C^n{}_{l,m}) \\
& + C_{kn,m} (6C_l^n{}_{,m} - 2C_l^m{}_{,n}) + 4C_{lm,k} \phi_{,m}{}^m - 2C^m{}_{m,k} \phi_{,l} - 2C_{kl,m} \phi_{,m} + 2\phi_{,k} \phi_{,l} + 4C_{km}{}^m (\phi_{,l} - C_{ln}{}^n) \\
& + 2(B_{l,k} - C_{kl}') (B^m{}_{,m} + \phi' + 4\mathcal{H}\phi) + 4\phi (\phi_{,kl} + B_{l,k}' - C_{kl}'') + B_{m,k} (B_l{}^m - \frac{3}{2} B^m{}_{,l}) \\
& + 2B_k (2C_{lm}{}^k - 2C^{m'}{}_{m,l} + B^m{}_{,lm} - \nabla^2 B_l - 2\mathcal{H}(B_l' + C^m{}_{m,l} - 2C_{lm}{}^m - \phi_{,l})) - \frac{3}{2} B_{k,m} B_l{}^m \Big] \\
& + C^{k'}{}_k \Big[2C^{ml} (C_{ml}' - B_{l,m}) + \phi C_{,l}^{l'} - 8\mathcal{H}\phi^2 - 4\phi\phi' - 2\phi B_{,l}^l + B^l (B_l' + C^m{}_{m,l} - 2C_{lm}{}^m - \phi_{,l} + 2\mathcal{H}B_l) \Big] \\
& + B^k \Big[B_k (C_{,l}^{l'} - B_{,l}^{l'} - \nabla^2 \phi - 2\mathcal{H}(2\phi' + B_{,l}^l)) + B^l (2C_{lm,k}{}^m - C^m{}_{m,kl} - \nabla^2 C_{kl} - 2\mathcal{H}(C_{kl}' + B_{l,k})) \\
& - \left(\frac{2a''}{a} - \mathcal{H}^2 \right) (2C_{kl} + 4\phi) + 2C^{lm} (2C_{km,l}' - 2C_{lm,k}' + B_{m,lk} - B_{k,ml} - 2\mathcal{H}(C_{lm,k} - 2C_{km,l})) \Big] \\
& - 4\phi^2 \Big[C_{,k}^{k''} - B_{,k}^{k'} - \nabla^2 \phi + 2\mathcal{H}(\mathcal{H}\phi - 3\phi' - B_{,k}^k) \Big] + C^{kl'} \phi (2B_{l,k} - 3C_{kl}') + \frac{1}{2} B_{l,k} \phi (B_{,l}^k - 3B_{,k}^l) \\
& + B_{,k}^k \phi (4\phi' + B_{,l}^l) + 2\phi (C_{kl}{}^k \phi_{,l} - C_{l,k}^l \phi_{,k} + 2\phi_{,k} \phi_{,k}) + 2B^{k'} (2C_{kl}{}^l \phi - \phi C_{,l}^l) \\
& + B^k \Big[C^m{}_{m,l} (B_{k,l} - B_{l,k} - 2C_{kl}') + C^{lm}{}_{,m} (4C_{kl}' - 2B_{k,l}) + B_{,l}^l (2C_{km}{}^m - C^m{}_{m,k} - B_k' + \phi_{,k}) \\
& + B_{m,l} (C_{,k}^l{}^m - 3C_{,k}^m{}_{,l} + C^{lm}{}_{,k}) + C_{lm}' (C_{,k}^m{}_{,l} - C^{lm}{}_{,k}) - \phi' (C_{l,k}^l - 2C_{kl}') + B_{,k}^l C_{ml}{}^m \\
& + 2\phi (2C_{kl}{}^{l'} - 2C_{l,k}^{l'} + B_{l,k}^l - \nabla^2 B_k - 4\mathcal{H}(2B_k' + C_{l,k}^l - 2C_{kl}{}^l - 2\phi_{,k})) \Big]. \tag{A4}
\end{aligned}$$

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